Answers to Problems found in Catalog Course Description for
Competitive Math for High School Students

Example 1:
How many zeros does 2009! end in ?

Solution:
We need to find out how many factors 10 there are in 2009!. As the prime factorization of 10 is $2 \cdot 5$ and since there are clearly more factors 2 than 5, we only need to count the number of factors 5. That number is given by Legendre's formula as
[2009/5] + [2009/25] + [2009/125] + [2009/625] + [2009/3125] + … =
401+80+16+3 + 0 + … = 500. So, 2009! ends in exactly 500 zeros. [From Chapter 6 (Number Theory I): This example uses something called Legendre's formula which will be discussed in the notes. It is about the number of trailing zeros of 2009 factorial].

Example 2:
The increasing sequence 2, 3, 5, 6, 7, 10, 11, … consists of all positive integers that are neither the square nor the cube of a positive integer. Find the 500^{th} term of the sequence.

Solution:
Because there aren't that many perfect squares or cubes, let's look for the smallest perfect square greater than 500. This happens to be $23^2 = 529$. Notice that there are 23 squares and 8 cubes less than or equal to 529, but 1 and 2^6 are both squares and cubes. Thus, there are 529-23-8+2=500 numbers in our sequence less than 529. Magically, we want the 500^{th} term, so our answer is the smallest non-square and non-cube less than 529, which is 528. [From Chapter 10 (Sequences)].

Example 3:
The surface of a soccer ball consists of only black regular pentagons and white regular hexagons. The edges of all pentagons border only on hexagons while the edges of the hexagons border alternately on pentagons and hexagons. If exactly 3 edges meet at each vertex, determine the number of pentagons and hexagons on the soccer ball.

Solution:
Let p and h be the number of pentagons and hexagons respectively. Furthermore let e be the number of vertices and k be the number of edges. Then Euler's formula says:
e + p + h = k + 2. Exactly two faces meet at each edge, hence the number of faces is twice the number of edges: 5p + 6h = 2e. Since by the condition of the problem every pentagon borders on 5 hexagons and every hexagon borders on 3 pentagons, we get 5p = 3h. Substituting this into the first two equations yields 5e + 8h = 5k + 10 and 9h = 2k. Eliminating h from these equations gives 45e = 90 + 2k. Finally since, also by the condition of the problem, exactly 3 edges meet at each vertex, we have 2k = 3e. Substituting this into the last equation gives k = 90. Then it follows easily that e = 60,
p = 12 and h = 20. So, the soccer ball consists of 12 pentagons and 20 hexagons. [From Chapter 9 (Combinatorial Geometry): This example uses Euler's polyhedron formula which is also discussed in the notes. It is about the surface of a soccer ball consisting only of regular pentagons and regular hexagons. Most soccer balls are indeed of this type: check your garage].